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PERTURBATION BOUNDS FOR THE DEFINITE GENERALIZED EIGENVALUE PRO--ETC(U)
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Perturbation Bounds for the Definite Generalized Eigenvalue Problem,

by

10 G. W. Stewart*

Abstract

Let A and B be Hermitian matrices and let $c(A,B) = \inf\{|x^H(A+iB)x| : \|x\|=1\}$. The eigenvalue problem $Ax = \lambda Bx$ is called definite if $c(A,B) > 0$. It is shown that a definite problem has a complete system of eigenvectors and its eigenvalues are real. Under perturbations of A and B, the eigenvalues behave like the eigenvalues of a Hermitian matrix in the sense that there is a 1-1 pairing of the eigenvalues with the perturbed eigenvalues and a uniform bound for their differences (in this case in the chordal metric). Perturbation bounds are also developed for eigenvectors and eigenspaces.

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Perturbation Bounds for the Definite Generalized Eigenvalue Problem

G. W. Stewart

1. Introduction

In this paper we shall be concerned with deriving perturbation bounds for the eigenvalues and eigenvectors of the generalized eigenvalue problem

$$Ax = \lambda Bx , \tag{1.1}$$

where A and B are Hermitian matrices of order n. When B is positive definite, as it is in most applications, the problem can be reduced to a Hermitian eigenvalue problem of the form

$$B^{-1/2}AB^{-1/2}y = \lambda y , \tag{1.2}$$

where $B^{1/2}$ is the positive definite square root of B. Thus, in this case, the eigenvalues are real.

When A and B are replaced by $\tilde{A} = A + E$ and $\tilde{B} = B + F$, the eigenvalues and eigenvectors will be perturbed by quantities that are functions of E and F. In principle one can apply the existing theory for Hermitian eigenvalue problems to (1.2) and obtain bounds on these perturbations; however, this approach is unsatisfactory for a number of reasons, which we shall now sketch (for a more complete discussion and examples see [3,10,12]).

When B is ill conditioned, that is when B is relatively near a singular matrix, the eigenvalues of (1.1) will typically be spread out, with some small and some large. The small ones may be relatively insensitive to perturbations in A and B. However, since the problem (1.2) has large

eigenvalues, $B^{-1/2}AB^{-1/2}$ must also be large, and so will its corresponding perturbation. The perturbation theory for (1.2) will then predict large perturbations for all the eigenvalues, even the small ones. A second difficulty is that the perturbation in B must be restricted so that B remains positive definite, even though perturbations that make B indefinite may have little effect on some of the eigenvalues. Finally, although the large eigenvalues of the problem usually undergo large perturbations, their reciprocals will undergo only small perturbations. This suggests that the usual Euclidean metric on the line is not appropriate for reporting the sizes of the perturbations in the eigenvalues.

In [12] the author has developed a perturbation theory for the non-Hermitian generalized eigenvalue problem that circumvents these difficulties, first by avoiding the use of inverses and second by using the chordal metric on the Riemann sphere (cf. Section 3). In [3] Crawford has described a class of Hermitian problems, called definite problems in this paper, that admit of a nice perturbation theory in the chordal metric. It is the purpose of this paper to strengthen and extend Crawford's results. In particular we shall obtain the one-one pairing of eigenvalues with their perturbed counterparts that holds for the Hermitian eigenvalue problem. In addition we shall obtain perturbation bounds for eigenspaces that are related to the perturbation bounds of Davis and Kahan [4] for invariant subspaces of Hermitian operators.

Throughout this paper A and B will be Hermitian matrices of order n, and

$$\tilde{A} = A + E, \quad \tilde{B} = B + F$$

will be Hermitian perturbations of A and B. The Euclidean vector norm and the spectral matrix norm will both be denoted by $\|\cdot\|$.

2. The geometric theory

Recently there has emerged an elegant geometric theory for the real symmetric generalized eigenvalue problem that is based on the quadratic form $x^T(A+iB)x$. For topological reasons, this theory does not apply when $n = 2$. In this section we shall extend this theory to the Hermitian problem in such a way that the restriction $n \neq 2$ is removed. We shall return to the real case at the end of the section.

Our ultimate goal is to replace the generalized eigenvalue problem (1) by an equivalent problem in which B is positive definite. Specifically for any real φ let

$$A_\varphi = A \cos \varphi - B \sin \varphi ,$$
$$B_\varphi = A \sin \varphi + B \cos \varphi .$$

Then it is obvious that any matrix X for which $X^H AX$ and $X^H BX$ are diagonal also diagonalizes A_φ and B_φ. Thus (1.1) has a complete set of eigenvectors if and only if the problem $A_\varphi x = \lambda B_\varphi x$ has the same set. We shall attempt to choose φ so that B_φ is positive definite. The condition under which this can be done is that the pair (A,B) be definite in the following sense.

Definition 2.1. The pair of Hermitian matrices (A,B) is definite if

$$c(A,B) \equiv \inf_{\|x\|=1} \{|x^H(A+iB)x|\} > 0 . \tag{2.1}$$

The eigenvalue problem (1.1) is definite if (A,B) is definite.

The proof that $c(A,B) > 0$ implies that B_φ is positive definite for some φ involves the geometry of the set over which the infimum in (2.1) is taken. Let

$$V = \{x^H(A+iB)x : \|x\|=1\}$$

and

$$C = \{x^H(A+iB)x : x \in C^n\}$$

$$\equiv \{\alpha v : v \in V, \alpha \geq 0\} .$$

Then $c(A,B) = \inf\{|v| : v \in V\}$. Moreover, C is the pointed cone generated by the closed, bounded set C. We may now state our fundamental theorem.

Theorem 2.2. The cone C is convex. If (A,B) is definite, then C lies properly in a closed half plane passing through the origin. Moreover there is a real number φ such that B_φ is positive definite and

$$c(A,B) = \lambda_{min}(B_\varphi) ,$$

where $\lambda_{min}(B_\varphi)$ denotes the smallest eigenvalue of B_φ.

Proof. The set V is just the field of values of $A + iB$, which is known to be convex (e.g. see [7]). Hence the cone C generated by V is convex. Now suppose that $c(A,B) > 0$. Let the infimum in (2.1) be

attained at x_0 and let $v_0 = x_0^H(A+iB)x_0$. Since V is convex $v_0 \in V$ satisfies $|v_0| = \inf\{|v| : v \in V\} = c(A,B) > 0$, V must lie in the closed half plane $H = \{z : R(\bar{v}_0 z) \geq |v_0|^2\}$, which does not contain the origin. Since V is bounded, it must be contained in a cone that is properly contained in the half plane $H - v_0$, which passes through the origin.

To prove the last part of the theorem, let V_φ, C_φ, and H_φ denote the field of values, the cone, and the half plane associated with the pair (A_φ, B_φ). Since

$$A_\varphi + iB_\varphi = e^{i\varphi}(A+iB) ,$$

it follows that V_φ, C_φ, and H_φ are just V, C, and H rotated clockwise through the angle φ. Choose φ so that H_φ lies in the upper half plane. Then $x_0^H A\, x_0 = 0$. Moreover for $\|x\| = 1$, $x^H B_\varphi x \geq c(A_\varphi, B_\varphi)$ $= c(A,B)$. Hence

$$0 < c(A,B) = x_0^H B\, x_0 = \inf_{\|x\|=1} x^H Bx .$$

This shows that B is positive definite, x_0 is an eigenvector of B_φ corresponding to $\lambda_{min}(B_\varphi)$, and $c(A,B) = \lambda_{min}(B_\varphi)$.□

Since in the above B_φ is positive definite, the Hermitian matrix $B_\varphi^{-1/2} A_\varphi B_\varphi^{-1/2}$ exists and can be diagonalized by a unitary matrix Y. It is easily verified that $X = B_\varphi^{-1/2} Y$ diagonalizes both A_φ and B_φ, and hence A and B. This proves

Corollary 2.3. If the problem (1.1) is definite, then there is a nonsingular matrix X such that X^HAX and X^HBX are both diagonal.

If we set $X^HAX = \mathrm{diag}(\mu_1,\mu_2,\ldots,\mu_n)$ and $X^HBX = \mathrm{diag}(\nu_1,\nu_2,\ldots,\nu_n)$, then the eigenvalues of (1.1) are given by $\lambda_i = \mu_i/\nu_i$. This admits the possibility of infinite eigenvalues when $\nu_i = 0$; however, the indeterminant case $\mu_i = \nu_i = 0$ cannot occur in a definite problem.

The number $c(A,B)$ will play an important role in the perturbation theory of the next two sections. The following theorem shows that it does not vary wildly with perturbations in A and B. This result is required in Section 3, where it is the number $c(\tilde{A},\tilde{B})$ that appears in the bounds.

Theorem 2.4.

$$c(\tilde{A},\tilde{B}) \geq C(A,B) - [\|E\|^2+\|F\|^2]^{1/2} .$$

Proof.

$$\begin{aligned}
c(\tilde{A},\tilde{B}) &= \inf_{\|x\|=1} \{[x^H(A+E)x]^2 + [x^H(B+F)x]^2\}^{1/2} \\
&\geq \inf_{\|x\|=1} [(x^HAx)^2+(x^HBx)^2]^{1/2} - \sup_{\|x\|=1} [(x^HEx)^2+(x^HFx)^2]^{1/2} \\
&\geq c(A,B) - [\sup_{\|x\|=1} (x^HEx)^2 + \sup_{\|x\|=1} (x^HFx)^2]^{1/2} \\
&= c(A,B) - [\|E\|^2+\|F\|^2]^{1/2}. \square
\end{aligned}$$

We turn now to the case where A and B are real. It is natural in this case to restrict the vector x in (2.1) to lie in $\mathbb{R}^n$ and define

$$V_r = \{x^T(A+iB)x : \|x\|=1,\ x \in \mathbb{R}^n\}$$

and

$$c_r(A,B) = \inf\{|v| : v \in V_r\} .$$

Brickman [1] has shown that if $n \neq 2$, then $V_r = V$, and consequently $c_r(A,B) = c(A,B)$. It follows that Theorem 2.2 is valid for $n \neq 2$, and in particular we can restrict our attention to real vectors. That this happy state of affairs does not hold for $n = 2$ is shown by the example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

for which $c(A,B) = 0 < 1 = c_r(A,B)$. However, we have the following corollary of Theorem 2.2.

Corollary 2.5. For $n = 2$, if $c(A,B) > 0$ then $c_r(A,B) = c(A,B)$.

Proof. By Theorem 2.2 we may find a φ such that B_φ is positive definite and an eigenvector x of B_φ such that $\|x\|=1$ and $x^H B_\varphi x = c(A,B)$. Since $x^H A_\varphi x = 0$, A_φ is indefinite. Now if the eigenvalue $c(A,B)$ is of multiplicity one, then x must be a scalar multiple of a real vector y, which then satisfies $y^T A_\varphi y = 0$ and $y^T B_\varphi y = c(A,B)$. If $c(A,B)$ is of multiplicity two, then any nonzero vector is an eigenvector. Since A_φ is real and indefinite, there is a vector y of norm unity such that $y^T A_\varphi y = 0$ and $y^T B_\varphi y = c(A,B)$.□

For the real case, much of the material in this section is implied by earlier work [1,2,3,5,6,14]. Hestenes [6] has characterized C as in

Theorem 2.2, and Crawford [3] appears to be the first to introduce the rotated problem. I am indebted to Hans Schneider for pointing out that the restriction $n \neq 2$ can be removed in the complex case. The characterization of $c(A,B)$ as the smallest eigenvalue of a positive definite B_φ is new and replaces an incorrect statement in Crawford's paper.

The number $c(A,B)$ appears repeatedly in the bounds to be derived in the next sections, which is why we have taken some pains to ascertain when the computationally simpler number $c_r(A,B)$ is equal to $c(A,B)$. Crawford was the first to realize the importance of the value of $c(A,B)$, as opposed to the relation $c(A,B) > 0$, and for this reason it is appropriate to refer to $c(A,B)$ as the Crawford number of the problem (1.1).

3. Perturbation bounds: eigenvalues

In this section we shall develop perturbation bounds for the eigenvalues of the definite generalized eigenvalue problem. For the Hermitian eigenvalue problem $Ax = \lambda x$, it is well known that if the eigenvalues are ordered so that $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$ and those of the perturbed problem $\tilde{A}\tilde{x} = \tilde{\lambda}\tilde{x}$ are ordered so that $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_n$ then

$$|\lambda_i - \tilde{\lambda}_i| \leq \|A - \tilde{A}\| \ , \quad (i=1,2,\ldots,n) \ .$$

We shall obtain a similar pairing of eigenvalues with their perturbed counterparts; however the ordering that defines the pairing must be defined in terms of certain angles associated with the eigenvalues of (1.1).

Let the pair (A,B) be definite and let $\|E\|^2 + \|F\|^2 < c(A,B)$, so that by Theorem 2.4 the pair $(\tilde{A},\tilde{B})$ is also definite. Let C and $\tilde{C}$ denote the cones associated with the two pairs. Then by Theorem 2.2 the complement of $C \cup \tilde{C}$ contains a ray R extending from the origin. For each nonzero point $u + iv \in \mathbb{C}$ define $\theta(u,v)$ as the angle subtended by R and $\{\alpha(u+iv) : \alpha \geq 0\}$ measured clockwise. By construction, θ is continuous on $C \cup \tilde{C}$.

Now let $Ax_i = \lambda_i Bx_i$, where $x_i \neq 0$. We define the angle associated with λ_i to be

$$\theta_i = \theta(x_i^H A x_i, x_i^H B x_i) ,$$

and throughout the rest of this section we shall assume that $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_n$. If B is positive definite, this corresponds to the natural ordering $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ of the eigenvalues. However, if B is not positive definite, this need not be. For example, if C lies in the right half plane then the positive eigenvalues will precede the negative ones.

The θ_i have a characterization that is completely analogous to the min-max characterization of the eigenvalues of the Hermitian problem.

<u>Theorem</u> 3.1. Let the problem (1.1) be definite and let its eigenvalues be ordered to that $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_n$. Then

$$\theta_i = \min_{\dim(X)=i} \max_{\substack{x \in X \\ x \neq 0}} \theta(x^T A x, x^T B x) \tag{3.1}$$

and

$$\theta_i = \max_{\dim(X)=n-i+1} \min_{\substack{x\in X\\ x\neq 0}} \theta(x^T Ax, x^T Bx) . \tag{3.2}$$

<u>Proof</u>. As in Section 2, choose φ so that B_φ is positive definite. Let the ray defining θ be rotated counterclockwise through φ and use this new ray to define a new function θ_φ. Then

$$\theta_\varphi(x^H A_\varphi x, x^H B_\varphi x) = \theta(x^H Ax, x^H Bx) ,$$

and without loss of generality we may drop the subscripts φ and assume that B is positive definite.

For B positive definite we have [7]

$$\lambda_i = \min_{\dim(X)=i} \max_{\substack{x\in X\\ x\neq 0}} \frac{x^H Ax}{x^H Bx} .$$

But because V is in the upper half plane, the mapping $\lambda \to \theta(\lambda,1)$ is increasing. Hence

$$\theta_i = \theta(\lambda_i,1) = \min_{\dim(X)=i} \max_{\substack{x\in X\\ x\neq 0}} \theta\left(\frac{x^H Ax}{x^H Bx}, 1\right)$$

$$= \min_{\dim(X)=i} \max_{\substack{x\in X\\ x\neq 0}} \theta(x^H Ax, x^H Bx) .$$

A proof of (3.2) follows from the characterization

$$\lambda_i = \min_{\dim(X)=n-i+1} \min_{\substack{x\in X\\ x\neq 0}} \frac{x^H Ax}{x^H Bx} . \square$$

As a consequence of Theorem 3.1, a number of separation theorems for the eigenvalues of symmetric matrices generalize to the definite problem (1.1). For example, if $\hat{A}$ and $\hat{B}$ are principal submatrices of A and B of order $n-1$, then the eigenvalues $\hat{\lambda}_1,\ldots,\hat{\lambda}_{n-1}$ of the problem $\hat{A}x = \hat{\lambda}Bx$ satisfy

$$\theta_1 \le \hat{\theta}_1 \le \theta_2 \le \hat{\theta}_2 \le \ldots \le \hat{\theta}_{n-1} \le \theta_n \ .$$

However, our main interest in the theorem is that it can be used to prove the following perturbation theorem.

<u>Theorem</u> 3.2. Let (A,B) be definite and let the eigenvalues of $Ax = \lambda Bx$ be ordered so that $\theta_1 \le \theta_2 \le \ldots \le \theta_n$. Let

$$\varepsilon = \{\|E\|^2+\|F\|^2\}^{1/2} ,$$

and assume that $\varepsilon < c(A,B)$ so that $(\tilde{A},\tilde{B}) = (A+E,B+F)$ is definite. Let the eigenvalues of $\tilde{A}\tilde{x} = \tilde{\lambda}\tilde{x}$ be ordered so that $\tilde{\theta}_1 \le \tilde{\theta}_2 \le \ldots \le \tilde{\theta}_n$. Then

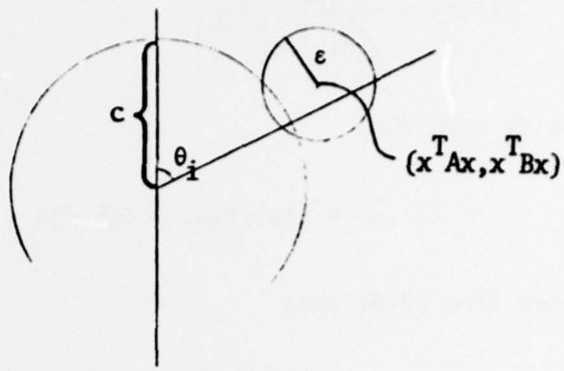
$$|\theta_i-\tilde{\theta}_i| \le \sin^{-1} \frac{\varepsilon}{c(A,B)} \ . \tag{3.3}$$

<u>Proof</u>. Let X_i be a minimizing subspace in the equality (3.1). Then

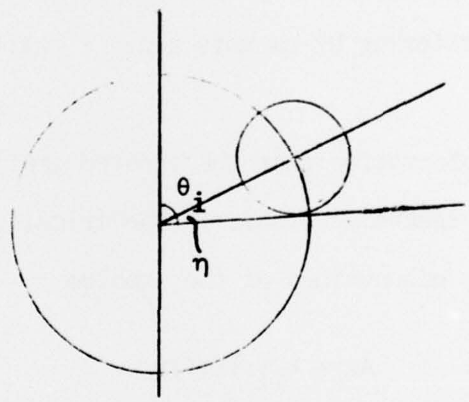
$$\hat{\theta}_i \le \max_{\substack{x\in X_i \\ x\neq 0}} \theta[x^T(A+E)x,\ x^T(B+F)x] \ . \tag{3.4}$$

Let $x \in X_i$ be a vector of norm unity for which the maximum in (3.4) is attained. Then since $\theta(x^TAx,x^TBx) \le \theta_i$, the point $(x^T(A+E)x,x^T(A+F)x)$

must lie in the circle of radius ε sketched below.



The maximum increase η of $\theta[x^T(A+E)x, x^T(B+F)x]$ over θ_i occurs when the circle is situated as shown below.



Elementary geometry now gives $\eta = \sin^{-1}[\varepsilon/c(A,B)]$, which shows that $\tilde{\theta}_i \le \theta_i + \eta$. The result $\theta_i - \eta \le \theta_i$ follows from a similar argument applied to the characterization (3.2) of θ_i.□

There are some observations to be made about the theorem. First, the bound (3.3) immediately implies a bound in the chordal metric. Specifically, let $\lambda = \mu/\nu$ and $\tilde{\lambda} = \tilde{\mu}/\tilde{\nu}$. Then the chordal distance between λ and $\tilde{\lambda}$ is

$$\chi(\lambda,\tilde{\lambda}) = \frac{|\mu\tilde{\nu} - \tilde{\mu}\nu|}{\sqrt{\mu^2+\nu^2}\,\sqrt{\tilde{\mu}^2+\tilde{\nu}^2}} \quad .$$

But it is easily seen that

$$\chi(\lambda,\tilde{\lambda}) = \sin|\theta(\mu,\nu)-\theta(\tilde{\mu},\tilde{\nu})| \ .$$

Then it follows from (3.3) that

$$\chi(\lambda_i,\tilde{\lambda}_i) \le \frac{\varepsilon}{c(A,B)} \quad .$$

Note that this inequality is somewhat weaker than (3.3), since eigenvalues that have angles differing by amounts near π will have a chordal difference near zero.

The *second observation* that the theorem implies the classical bounds for the Hermitian eigenvalue problem. Specifically let $B = I$, and let $\lambda_i(\tau) = \lambda_i/\tau$ be the eigenvalues of the problem

$$Ax = \lambda_i(\tau)(\tau I)x \ . \tag{3.5}$$

Let $c(\tau) = c(A,\tau I)$ be the Crawford number for the problem (3.5). Note that when τ is large, $c(\tau) = \tau + 0(1)$. If we consider a Hermitian perturbation in A of norm ε, then we have the bounds

$$|\theta_i(\tau)-\tilde{\theta}_i(\tau)| \le \sin^{-1}\frac{\varepsilon}{c(\tau)} = \frac{\varepsilon}{\tau} + 0\left(\frac{1}{\tau^2}\right). \tag{3.6}$$

Since $\lambda_i(\tau)$ and $\tilde{\lambda}_i(\tau)$ approach 0 as $1/\tau$, we have

$$(3.7) \qquad |\theta_i(\tau)-\tilde{\theta}_i(\tau)| = |\lambda_i(\tau)-\tilde{\lambda}_i(\tau)| + 0\left(\frac{1}{\tau^2}\right).$$

Combining (3.6) and (3.7) and multiplying by τ, we get

$$|\lambda_i-\tilde{\lambda}_i| \leq \varepsilon + 0\left(\frac{1}{\tau}\right),$$

which gives the classical result when $\tau \to \infty$.

Finally we note that it follows from the results in [12] that for a simple eigenvalue λ_i and ε sufficiently small,

$$\chi(\lambda_i,\tilde{\lambda}_i) \leq \frac{\varepsilon}{\sqrt{(x_i^H A x_i)^2+(x_i^H B x_i)^2}} + O(\varepsilon^2) \ .$$

Since $c^2(A,B) \leq (x_i^H A x_i)^2 + (x_i^H B x_i)^2$, it is seen that we pay a price in the sharpness of our bounds to gain freedom from considerations of multiplicity.

4. Perturbation bounds: eigenspaces

In this section we shall derive perturbation bounds for the eigenvectors of the definite problem (1.1). These bounds imply that eigenvectors corresponding to poorly separated eigenvalues are very sensitive to perturbations in A and B; however, the subspace spanned by the eigenvectors corresponding to a cluster of eigenvalues may be relative insensitive. We shall, therefore, phrase our bounds in terms of subspaces rather than individual eigenvectors. This approach is analogous to the one taken in the ordinary eigenvalue problem, where one bounds perturbations in invariant subspaces corresponding to clustered eigenvalues [4,8,9,11].

We begin by noting that if x is an eigenvector of (1.1), then Ax and Bx are dependent. Following [10], we can generalize this idea to subspaces.

Definition 4.1. A subspace $\mathcal{X}$ is an eigenspace of (1.1) if

$$\dim(A\mathcal{X}+B\mathcal{X}) \leq \dim(\mathcal{X}) .$$

Clearly any set of eigenvectors of (1.1) spans an eigenspace. Conversely, if the problem is definite, then an eigenspace is spanned by a set of eigenvectors. To see this, we first note that the space $A_\varphi\mathcal{X} + B_\varphi\mathcal{X}$, where A_φ and B_φ are defined as in Section 2, is the same as $A\mathcal{X} + B\mathcal{X}$. Hence we may assume that B is positive definite. Now let $\mathcal{X}_1$ be an eigenspace of dimension ℓ spanned by the columns of the full rank matrix X_1. Then $\dim(B\mathcal{X}_1) = \ell$, and $B\mathcal{X}_1$ has an orthogonal complement $\mathcal{X}_2$ of dimension $n-\ell$. Since B is positive definite $\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathbb{C}^n$. Let the columns of the $n \times (n-\ell)$ matrix X_2 span $\mathcal{X}_2$, so that $X_2^TBX_1 = 0$. But from the definition of eigenspace and the fact that $\dim(B\mathcal{X}_1) = \ell$, we have that $A\mathcal{X} \subset B\mathcal{X}$. Hence $X_2^TAX_1 = 0$. It follows that

$$(X_1,X_2)^TA(X_1,X_2) = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \tag{4.1}$$

and

$$(X_1,X_2)^TB(X_1,X_2) = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} . \tag{4.2}$$

Now the problems $M_iy = \lambda N_iy$ $(i=1,2)$ are definite; hence M_i and N_i can be

simultaneously diagonalized by nonsingular matrices Y_i. Then (X_1Y_1, X_2Y_2) is nonsingular and diagonalizes A and B. In particular the columns of X_1Y_1 are eigenvectors spanning $\mathcal{X}_1$.

Turning now to the perturbation theorem, we consider an eigenspace $\mathcal{X}_1$ of dimension ℓ of the definite problem (1.1). Let $\mathcal{X}_2$ be its complementary eigenspace. Let the columns of $X_1 = (x_1, x_2, \ldots, x_\ell)$ and $X_2 = (x_{\ell+1}, \ldots, x_n)$ be eigenvectors spanning $\mathcal{X}_1$ and $\mathcal{X}_2$, chosen so that M_i and N_i (i=1,2) in (4.1) and (4.2) are diagonal. Set

$$M_1 = \mathrm{diag}(\mu_1, \ldots, \mu_\ell) \qquad N_1 = \mathrm{diag}(\nu_1, \ldots, \nu_\ell)$$

$$M_2 = \mathrm{diag}(\mu_{\ell+1}, \ldots, \mu_n) \qquad N_2 = \mathrm{diag}(\nu_{\ell+1}, \ldots, \nu_n) .$$

We shall use the same notation for the perturbed eigenvalue problem $\tilde{A}\tilde{x} = \tilde{\lambda}\tilde{B}\tilde{x}$, except that all quantities will be overlined with tildas. We shall assume that both problems are definite.

We begin with a lemma which furnishes a possible basis for the perturbed eigenspace. We will obtain sharper results if we recognize that certain infima that are bounded below by Crawford numbers are actually taken over eigenspaces. Accordingly, we define

$$c(A,B;\mathcal{X}) = \inf \{|x^H(A+iB)x| \; : \; x \in \mathcal{X}, \|x\|=1\} .$$

<u>Lemma</u> 4.2. For i = 1,2,...,n let $\lambda_i = \mu_i/\nu_i$ and $\tilde{\lambda}_i = \tilde{\mu}_i/\tilde{\nu}_i$, and let

$$\delta = \min \{\mathcal{X}(\lambda_i, \tilde{\lambda}_j) \; : \; i=1,\ldots,\ell;\; j=\ell+1,\ldots,n\} .$$

If $\delta > 0$, then there is a matrix Q whose columns satisfy

$$\frac{\|q_i\|}{\|x_i\|} \le \frac{\varepsilon}{\chi(\tilde{A},\tilde{B};\tilde{X}_2)\delta}$$

such that

$$R(X_1 + Q) \subset \tilde{X}_1 ,$$

where $R(X)$ denotes the column space of X.

Proof. Let $\underset{\sim}{\tilde{B}}_2$ denote operator defined by restricting $\tilde{B}_2$ to the space $\tilde{X}_2$. By rotating the problem, we may assume that $\underset{\sim}{\tilde{B}}_2$ is positive definite and

$$\|\underset{\sim}{\tilde{B}}_2^{-1}\|^{-1} = c(\tilde{A},\tilde{B};X_2) .$$

Note that under this assumption N_2 is positive definite.

We shall seek Q in the form

$$Q = \tilde{X}_2 P .$$

From the definition of eigenspace $R(X_1+Q) \subset \tilde{X}_1$ if and only if $R[\tilde{B}(x_1+Q)] \perp \tilde{X}_2$. This is equivalent to requiring that $\tilde{X}_2^T\tilde{B}(X_1+\tilde{X}_2P) = 0$ or

$$\tilde{X}_2^T\tilde{B}X_1 = -\tilde{N}_2 P . \tag{4.3}$$

To develop an expression for $\tilde{X}_2^T\tilde{B}X_1$, note that since

$$(\tilde{A}-E)X_1N_1 = (\tilde{B}-F)X_1M_1 ,$$

we have

$$\tilde{X}_2^T A X_1 N_1 - \tilde{X}_2^T \tilde{B} X_1 = \tilde{X}_2^T (EXN_1 - FXM_1) \equiv \tilde{X}_2^T R \ .$$

But $\tilde{N}_2 \tilde{X}_2^T \tilde{A} = \tilde{M}_2 \tilde{X}_2^T \tilde{B}$, so that

$$\tilde{M}_2 \tilde{X}_2^T \tilde{B} X_1 N_1 - \tilde{N}_2 \tilde{X}_2^T \tilde{B} X_1 M_1 = \tilde{N}_2 \tilde{X}_2^T R \ .$$

Hence, because $\tilde{M}_2$ and $\tilde{N}_2$ commute, if we choose P to satisfy

$$\tilde{M}_2 P N_1 - \tilde{N}_2 P M_1 = -\tilde{X}_2^T R \ ,$$

then $N_2 P$ will satisfy (4.3).

Let $\tau_{ji} = \tilde{\mu}_j \nu_i - \tilde{\nu}_j \mu_i$. Then the hypothesis $\delta > 0$ implies that $\tau_{ji} \neq 0$ $(i=1,\ldots,\ell; j=\ell+1,\ldots,n)$. Consequently P is uniquely defined and its $(j-\ell,i)$ element is given by

$$p_{j-\ell,i} = \frac{\tilde{x}_j^T r_i}{\tau_{ij}} \ ,$$

where r_i denotes the i-th column of R. It follows that the i-th column of $Q = \tilde{X}_2 P$ is given by

$$q_i = \left(\sum_{j=\ell+1}^{n} \frac{\tilde{x}_j \tilde{x}_j^T}{\tau_{ij}} \right) r_i \equiv S_i r_i \ ,$$

and our problem is reduced to determining bounds on $\|S_i\|$.

Let $\tilde{Y}_2 = (\tilde{y}_{\ell+1}, \tilde{y}_{\ell+2}, \ldots, \tilde{y}_n) = \tilde{X}_2 \tilde{N}_2^{-1/2}$. Then

$$S_i = \sum_{j=\ell+1}^{n} \frac{\nu_j}{\tau_{ij}} \tilde{y}_j \tilde{y}_j^T \ .$$

But if $\tilde{B}_2$ is the operator defined at the beginning of the proof

$$\tilde{Y}_2^T\tilde{B}_2\tilde{Y}_2 = \tilde{N}_2^{-1/2}\tilde{X}_2^T\tilde{B}_2\tilde{X}_2\tilde{N}_2^{-1/2} = I ,$$

so that the columns of $\tilde{B}_2^{1/2}\tilde{Y}_2$ are orthonormal. Hence the eigenvalues of $\tilde{B}_2^{1/2}S_i\,\tilde{B}_2^{1/2}$ are $\tilde{\nu}_j/\tilde{\tau}_{ij}$. Hence

(4.4) $$\|S_i\| = \|\tilde{B}_2^{-1/2}(\tilde{B}_2^{1/2}S_i\tilde{B}_2^{1/2})\tilde{B}_2^{-1/2}\| \le \|\tilde{B}_2^{-1}\|\|\tilde{B}_2^{1/2}S_i\tilde{B}_2^{1/2}\|$$

$$\le \frac{\max_j\{|\tilde{\nu}_j/\tau_{ij}|\}}{\tilde{c}_2} \le \frac{\max_j\{|\sqrt{\tilde{\mu}_j^2+\tilde{\nu}_j^2}/\tau_{ij}|\}}{\tilde{c}_2}$$

where $\tilde{c}_2 = c(\tilde{A},\tilde{B};\tilde{X}_2)$.

Now from (4.5)

(4.5) $$q_i = S_i r_i = S_i(Ex_i\nu_i+Fx_i\mu_i) .$$

Hence

(4.6) $$\|q_i\| \le \|S_i\|\|x_i\|\varepsilon\sqrt{\mu_i^2+\nu_i^2} .$$

Combining (4.5) and (4.6) we see that

$$\frac{\|q_i\|}{\|x_i\|} \le \frac{\varepsilon}{\tilde{c}_2}\max_j \frac{\sqrt{\mu_i^2+\nu_i^2}\sqrt{\tilde{\mu}_j^2+\tilde{\nu}_j^2}}{|\mu_i\tilde{\nu}_j-\nu_i\tilde{\mu}_j|} \le \frac{\varepsilon}{\tilde{c}_2\delta} . \square$$

The bound in the theorem may be written in the form

$$\frac{\|Q\|_F}{\|X_1\|_F} \le \frac{\varepsilon}{\tilde{c}_2\delta}$$

where $\|X\|_F^2 \equiv \Sigma|x_{ij}|^2$ denotes the Frobenius norm of X. Thus the bound depends directly on the perturbation and inversely on the gap between the eigenvalues, with the Crawford number determining the size of the effect. Unfortunately $R(X_1+Q_1)$ need not be an eigenspace unless $\dim [R(X_1+Q_1)] = \ell$. When $\ell = 1$, so that X_1 is the single eigenvector x_1, we can assure this by requiring that $\|q_1\| < \|x_1\|$.

Theorem 4.3. Let $\delta = \min \{\chi(\lambda_1,\tilde{\lambda}_j) : j=2,3,\ldots,n\}$. If $\varepsilon/\delta < c(\tilde{A},\tilde{B};X_2)$, then there is a vector q_1 satisfying

$$\frac{\|q_1\|}{\|x_1\|} \le \frac{\delta}{\delta c(\tilde{A},\tilde{B};X_2)} < 1$$

such that $x_1 + q$ is an eigenvector of $\tilde{A}\tilde{x} = \tilde{\lambda}\tilde{B}\tilde{x}$ corresponding to $\tilde{\lambda}_1$.

When $\ell < 1$, we must take into account the effect of near dependencies among the columns of X. Define

$$\inf(X_1) = \inf_{\|x\|=1} \|X_1 x\| .$$

Then if $\|Q\| < \inf(X_1)$, $\mathrm{rank}(X+Q) = \mathrm{rank}(X) = \ell$. These considerations lead to the following theorem.

Theorem 4.4. Let

$$\eta = \frac{\varepsilon\sqrt{\ell}}{\delta c(\tilde{A},\tilde{B};\tilde{X}_2)} \left\{ \frac{\sqrt{\|A\|^2+\|B\|^2}}{c(A,B;X_1)} \right\}^{1/2} .$$

If $\eta < 1$, then $R(X_1+Q) = \tilde{X}_1$.

Proof. It follows from the proof of Theorem 4.3 that if x_j is scaled by a factor α, then q_j is scaled by the same factor. Hence we may assume that $\|x_j\| = 1$. We first find a lower bound on $\inf(X_1)$. By rotating (A,B), we may assume that if $\underset{\sim}{B}_1$ denotes the restriction of B to X_1 then $\underset{\sim}{B}_1$ is positive definite and $\inf(\underset{\sim}{B}_1) = c(A,B;X_1)$. Of course,

$$\|\underset{\sim}{B}_1\| \le \|B\| \le \sqrt{\|A\|^2+\|B\|^2} \; . \tag{4.7}$$

Now

$$X_1^T \underset{\sim}{B}_1 X_1 = N_1,$$

from which it follows that the columns of $U = \underset{\sim}{B}_1^{1/2} X_1 N_1^{-1/2}$ are orthonormal. Hence

$$1 = \inf(U) \le \|\underset{\sim}{B}_1^{1/2}\| \|N_1^{-1/2}\| \inf(X_1) \; .$$

Since the columns of X_1 have norm unity, we have $\nu_i \ge c(A,B;X_1)$. Hence from (4.7)

$$\inf(x) \ge \left\{ \frac{c(A,B;X_1)}{\sqrt{\|A\|^2+\|B\|^2}} \right\}^{1/2} \; . \tag{4.8}$$

But

$$\|Q\|_F \le \frac{\varepsilon \|X\|_F}{\delta \tilde{c}_2} \le \frac{\varepsilon \sqrt{\ell}}{\delta \tilde{c}_2} \; , \tag{4.9}$$

and the result follows from (4.9) and (4.10).□

When $\ell = 1$, Theorem 4.5 does not reduce to Theorem 4.3 because we have given too much away in the bound (4.9). The principal application is to the case $\ell > 1$, where the condition $\eta < 1$ not only guarantees that $\tilde{X}_1 = R(X_1+Q)$ but also implies that X_1 and $\tilde{X}_1$ are acutely situated with respect to one another. In fact a slight modification of the proof of Theorem 4.1 in [13] gives the following corollary.

Corollary 4.5. Let P_{X_1} and $P_{\tilde{X}_1}$ denote the orthogonal projections onto X_1 and $\tilde{X}_1$. If $\eta < 1$, then

$$\|P_{X_1} - P_{\tilde{X}_1}\|_F \leq \frac{\eta}{1-\eta} .$$

It is unfortunate that η contains the factor $\sqrt{\ell}$, since it grows with dimension of X_1. The presence of this factor is a direct consequence of the fact that Lemma 4.2 bounds $\|Q\|_F/\|X\|_F$ instead of $\|Q\|/\|F\|$. For the Hermitian eigenvalue problem, Davis and Kahan [4] have been able to obtain bounds in the spectral norm imposing additional restrictions on the location of the eigenvalues. Whether such bounds can be obtained for the definite generalized eigenvalue problem is an open question.

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Let A and B be Hermitian matrices and let $c(A,B) = \inf\{|x^H(A+iB)x| : \|x\|=1\}$. The eigenvalue problem $Ax = \lambda Bx$ is called definite if $c(A,B) > 0$. It is shown that a definite problem has a complete system of eigenvectors and its eigenvalues are real. Under perturbations of A and B, the eigenvalues behave like the eigenvalues of a Hermitian matrix in the sense that there is a 1-1 pairing of the eigenvalues with the perturbed eigenvalues and a uniform bound for their differences (in this case in the chordal metric). Perturbation bounds are also developed for eigenvectors and eigenspaces.

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